JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 118 (2000) 311–322

Entropic integrals of orthogonal hypergeometric polynomials with general supports

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Received 15 March 1999

Abstract

The Boltzmann–Shannon information entropy of probability measures which involve the continuous hypergeometric-type polynomials $\{p_n(x)\}$, orthogonal with respect to a general weight function $\omega(x)$, is determined by two integral quantities: one with kernel $p_n^2(x)\omega(x)\ln p_n^2(x)$, called as entropy of the polynomial $p_n(x)$, and another one with kernel $p_n^2(x)\omega(x)\ln\omega(x)$. Here, an explicit expression for the latter quantity, and for a broader family of related integrals, is obtained in terms only of the second-order differential equation satisfied by the involved polynomials. For illustration, the general formula is applied to evaluate the integrals corresponding to the three classical families of continuous orthogonal polynomials on the real axis of hypergeometric type (Hermite, Laguerre, and Jacobi). © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Hypergeometric polynomials; Information entropy; Second-order differential equations; Probability measures; Entropy-like integrals

1. Introduction

Let $\{p_n(x)\}$ denote a sequence of real polynomials orthogonal with respect to the weight function $\omega(x)$ on the interval $[a,b]\subseteq\mathbb{R}$,

$$\int_a^b p_n(x)p_m(x)\omega(x)\,\mathrm{d}x=\varkappa_n\delta_{n,m} \tag{1}$$

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PII: S0377-0427(00)00296-X

with $\deg p_n(x) = n$. Let us also assume that the weight function $\omega(x)$ is nonnegative definite, i.e., $\omega(x) \geqslant 0 \ \forall x \in [a, b]$. The functions $f_n(x)$ defined as

$$f_n(x) = \frac{1}{\varkappa_n} p_n^2(x)\omega(x) \tag{2}$$

can then be considered as normalized density functions for the continuous random variable x. The spread or extent of such a probability distribution is measured by its Boltzmann–Shannon information entropy

$$S_n = -\int_a^b f_n(x) \ln f_n(x) \, \mathrm{d}x, \tag{3}$$

which according to Shannon's information theory [20] is the only rigorous measure of uncertainty for a continuous random variable whose density function is $f_n(x)$. Substituting Eq. (2) into (3), the latter equation can be written as

$$S_n = \ln \varkappa_n + \frac{1}{\varkappa_n}(E_n + I_n), \tag{4}$$

where

$$E_n = -\int_a^b p_n^2(x)\omega(x) \ln p_n^2(x) \, \mathrm{d}x, \tag{5}$$

and

$$I_n = -\int_a^b p_n^2(x)\omega(x) \ln \omega(x) \, \mathrm{d}x. \tag{6}$$

Consider the second-order differential operator

$$\mathscr{F}[y](x) = \sigma(x)y''(x) + \tau(x)y'(x), \tag{7}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials whose degrees are not greater than 2 and 1, respectively. If $|\tau'| + |\sigma''| \neq 0$, then for every $n \in \mathbb{N}$, $\mathscr{F}$ has a polynomial eigenfunction $y = y_n(x)$ of degree n corresponding to the eigenvalue

$$\lambda_n = n\tau' + \tfrac{1}{2}n(n-1)\sigma''. \tag{8}$$

These polynomials are usually called *hypergeometric-type polynomials* (or *continuous hypergeometric-type polynomials*, in contrast with those which appear as eigenfunctions of second order linear *difference* operators). They can be reduced by means of linear changes of the variable to one of the four classical families, i.e., Hermite, Laguerre, Jacobi and Bessel [17]. The hypergeometric-type polynomials appear in the mathematical modelling of a great amount of physical and chemical phenomena (see e.g. [5–7,10,16,17,23,28]). In particular, the quantum–mechanical equation of motion in position or momentum space of numerous many-particle systems can be very often reduced to a second-order differential equation of hypergeometric character [6,10,17]. In this case, the modulus squared of the normalized wave functions corresponding to the bound states of the system have the form of the functions $f_n(x)$ defined in Eq. (2), and the determination of the Boltzmann–Shannon information entropy S_n measuring the uncertainty about the outcome of a measurement of position or momentum in the nth stationary state reduces to the evaluation of the entropic integrals E_n and I_n of the associated orthogonal polynomials of hypergeometric type.

The calculation of the integrals E_n, which are often called "entropies of the orthogonal polynomials $p_n(x)$" has been recently the subject of considerable effort for both fundamental and applied reasons [1–3,8,9,12,13,18,21,25–27]. These integrals are closely related to the L^p-norms, whose study is of independent interest in the theory of general orthogonal and extremal polynomials [2]. Asymptotic formulas for E_n in the $n \to \infty$ limit have been obtained for general orthogonal polynomials on a finite interval [1–3,26], Laguerre polynomials [2,13,26], and Freud polynomials, which are orthogonal on the whole real axis with $\omega(x)=\exp(-|x|^m)$, $m > 0$, and include Hermite polynomials as the particular case $m = 2$ [2,3,21,26]. The problem of determining the exact analytical value of E_n is generally very difficult, although it has been solved for Chebyshev polynomials of the first and second kinds [12,25,26] and some progress has been achieved for general Gegenbauer polynomials [8,9,12,26,27] and Hermite [18] polynomials. For a general and updated survey of this topic, see [14].

In this paper we shall focus on the other quantity which is required, together with E_n, to determine the Boltzmann–Shannon entropy S_n of the probability density $f_n(x)$: the integral I_n defined by Eq. (6). The exact analytical value of this quantity is known for the Gegenbauer [22,26,27], Freud [21,26], and Laguerre polynomials [13,26], while its asymptotic behavior is known for the polynomials orthogonal with respect to a weight function supported on the interval $[-1, 1]$ [1]. Our aim here is to find in a unified way the analytical expression of the integrals I_n for orthogonal polynomials of hypergeometric type with both compact and noncompact support. To do this, we begin by collecting, in Section 2, some basic knowledge about the hypergeometric-type polynomials which is used later on. Then, in Section 3, a recursion relation and a closed analytical expression for a broad family of integrals which includes I_n as a particular case are given, in terms *only* of the coefficients of the differential equation satisfied by the polynomials $p_n(x)$. In Section 4, we apply our general results to evaluate I_n for the three classical families of hypergeometric-type orthogonal polynomials on the real axis (Hermite, Laguerre and Jacobi). Finally, in Section 5, some summarizing and concluding remarks are given.

2. General properties of orthogonal hypergeometric polynomials

We introduce first some notation. If $\{y_n(x)\}_{n\in\mathbb{N}}$ stands for the sequence of monic hypergeometric polynomials corresponding to operator (7), then we denote by $y_{n,k}(x)$, for $n, k \in \mathbb{N}$, the monic polynomial eigenvalue of degree n of the operator

$$\mathscr{F}_k[y](x) = \sigma(x)y''(x) + \tau_k(x)y'(x), \quad \tau_k(x) = \tau(x) + k\sigma'(x) \tag{9}$$

with $|\tau_k'| + |\sigma''| \neq 0$, so that

$$y_n(x) = y_{n,0}(x) \quad \text{and} \quad y_{n,k}'(x) = n y_{n-1,k+1}(x). \tag{10}$$

Analogous notation will be used for other sequences of monic hypergeometric polynomials as well. For the sake of brevity, below we omit the subindex k when $k = 0$.

An explicit expression for these polynomials is provided by the Rodrigues' formula. Fix

$$\omega(x) = \exp \int^x \frac{\tau(t) - \sigma'(t)}{\sigma(t)}\, \mathrm{d}t, \quad \omega_k(x) = \sigma^k(x)\omega(x), \quad k \geqslant 0, \tag{11}$$

then $\omega_k(x)$ is a solution of the so-called Pearson's equation, $[\sigma(x)\omega_k(x)]' = \tau_k(x)\omega_k(x)$. If

$$A_{n,k} = \prod_{j=0}^{n-1}\left[\tau_k' + \frac{n+j-1}{2}\sigma''\right]^{-1} = \prod_{j=0}^{n-1}\left[\tau' + \frac{n+j+2k-1}{2}\sigma''\right]^{-1}, \tag{12}$$

then

$$y_{n,k}(x) = \frac{A_{n,k}}{\omega_k(x)}\frac{\mathrm{d}^n\omega_{n+k}(x)}{\mathrm{d}x^n}. \tag{13}$$

In what follows, we assume additionally the existence of two values $-\infty \leqslant a < b \leqslant +\infty$ such that

$$\omega(x) \in C(a,b) \quad \text{and} \quad \lim_{x\to a+}\omega_1(x)x^k = \lim_{x\to b-}\omega_1(x)x^k = 0, \quad k \geqslant 0. \tag{14}$$

This assumption leads to orthogonality of $\{y_n(x)\}$ with respect to the weight function $\omega(x)$ on the interval $[a,b]$ (see [17]),

$$\int_a^b y_n(x)y_m(x)\omega(x)\,\mathrm{d}x = \varkappa_n\delta_{n,m}, \qquad \varkappa_n = (-1)^n n!\,A_n\gamma_n, \tag{15}$$

where $\{\gamma_n\}$ denotes the sequence of generalized moments of $\omega(x)$,

$$\gamma_n = \int_a^b \omega_n(x)\,\mathrm{d}x. \tag{16}$$

Likewise, every sequence $\{y_{n,k}(x)\}$ is then orthogonal with respect to the weight function $\omega_k(x)$ on the same interval,

$$\int_a^b y_{n,k}(x)y_{m,k}(x)\omega_k(x)\,\mathrm{d}x = \varkappa_{n,k}\delta_{n,m}, \qquad \varkappa_{n,k} = (-1)^n n!\,A_{n,k}\gamma_{n+k}. \tag{17}$$

3. General results

With the notations introduced in the previous sections, let us consider the family of integrals

$$I(n,k,l) = -\int_a^b [y_{n,k}(x)]^2\omega_k(x)\ln\omega_l(x)\,\mathrm{d}x, \tag{18}$$

which includes I_n as the particular case $k = l = 0$, $I_n = I(n,0,0)$. In this section we aim at finding a recurrence relation and a closed analytical formula for $I(n,k,l)$ in terms of the polynomials $\sigma(x)$ and $\tau(x)$ characterizing the second-order differential operator (7) whose polynomial eigenfunctions are the $y_n(x) = y_{n,0}(x)$. Our results are summarized in the following theorems.

Theorem 1. *If the hypergeometric-type polynomials $\{y_n(x)\}$ are eigenfunctions of the second-order differential operator defined in Eq.* (7), *then the integrals $I(n,k,l)$ in* (18) *satisfy the recurrence relation*

$$I(n,k,l) = \frac{A_{n,k}}{A_{n-1,k+1}}\{-nI(n-1,k+1,l) + [\tau' + (l-1)\sigma'']\varkappa_{n,k}\}, \tag{19}$$

where the constants $A_{n,k}$ and $\varkappa_{n,k}$ are defined in Eqs. (12) *and* (17), *respectively.*

Proof. Using Eq. (13), we readily find that

$$[y_{n,k}(x)\omega_k(x)]' = A_{n,k}\frac{\mathrm{d}^{n+1}\omega_{n+k}(x)}{\mathrm{d}x^{n+1}},$$

$$y_{n+1,k-1}(x)\omega_{k-1}(x) = A_{n+1,k-1}\frac{\mathrm{d}^{n+1}\omega_{n+k}(x)}{\mathrm{d}x^{n+1}}.$$

Combining these two equations, and replacing k by $k+1$ and n by $n-1$, we obtain

$$[y_{n-1,k+1}(x)\omega_{k+1}(x)]' = \frac{A_{n-1,k+1}}{A_{n,k}}y_{n,k}(x)\omega_k(x).$$

Substituting this expression into (18), we have

$$I(n,k,l) = -\frac{A_{n,k}}{A_{n-1,k+1}}\int_a^b [y_{n,k}(x)\ln\omega_l(x)][y_{n-1,k+1}(x)\omega_{k+1}(x)]'\,\mathrm{d}x,$$

which after integrating by parts with account of (14) reads

$$I(n,k,l) = \frac{A_{n,k}}{A_{n-1,k+1}}\int_a^b [y_{n-1,k+1}(x)\omega_{k+1}(x)][y_{n,k}(x)\ln\omega_l(x)]'\,\mathrm{d}x.$$

The derivative that appears in the last expression can be evaluated using Eqs. (10) and (11),

$$[y_{n,k}(x)\ln\omega_l(x)]' = n\,y_{n-1,k+1}(x)\ln\omega_l(x) + y_{n,k}(x)\left[\frac{\tau(x)+(l-1)\sigma'(x)}{\sigma(x)}\right].$$

We thus obtain

$$I(n,k,l) = \frac{A_{n,k}}{A_{n-1,k+1}}\left\{n\int_a^b [y_{n-1,k+1}(x)]^2\omega_{k+1}(x)\ln\omega_l(x)\,\mathrm{d}x\right.$$

$$\left. + \int_a^b [y_{n-1,k+1}(x)\omega_{k+1}(x)]y_{n,k}(x)\left[\frac{\tau(x)+(l-1)\sigma'(x)}{\sigma(x)}\right]\mathrm{d}x\right\},$$

which recalling definition (18) and Eq. (11) can be written as

$$I(n,k,l) = \frac{A_{n,k}}{A_{n-1,k+1}}\left\{-nI(n-1,k+1,l)\right.$$

$$\left. + \int_a^b y_{n,k}(x)y_{n-1,k+1}(x)\omega_k(x)[\tau(x)+(l-1)\sigma'(x)]\,\mathrm{d}x\right\}. \tag{20}$$

Since the degrees of $\sigma(x)$ and $\tau(x)$ are not greater than 2 and 1, respectively, these polynomials have the general forms

$$\sigma(x) = a + bx + cx^2, \quad \tau(x) = d + ex,$$

where a, b, c, d, e are constants. Taking the values of these polynomials and their derivatives at $x=0$, we can rewrite the previous formula as

$$\sigma(x) = \sigma(0) + \sigma'(0)x + \tfrac{1}{2}\sigma''x^2, \quad \tau(x) = \tau(0) + \tau'x.$$

Substituting these expressions into Eq. (20), we get

$$I(n,k,l)=\frac{A_{n,k}}{A_{n-1,k+1}}\left\{-nI(n-1,k+1,l)+[\tau(0)+(l-1)\sigma'(0)]\int_a^b y_{n,k}(x)y_{n-1,k+1}(x)\omega_k(x)\,\mathrm{d}x\right.$$

$$\left.+[\tau'+(l-1)\sigma'']\int_a^b y_{n,k}(x)[xy_{n-1,k+1}(x)]\omega_k(x)\,\mathrm{d}x\right\}. \tag{21}$$

The second term in the right-hand side of this formula vanishes because of the orthogonality of the sequence $\{y_{n,k}(x)\}$, since $y_{n-1,k+1}(x)$ is a polynomial of degree $n-1$ and therefore its expansion in series of the $\{y_{m,k}(x)\}$ can only contain terms such that $m<n$. Taking into account that we are working with monic polynomials, so that $xy_{n-1,k+1}(x)=y_{n,k}(x)+\mathrm{O}(x^{n-1})$, the orthogonality property (17) also enables us to simplify the third term in the right-hand side of (21). We thus obtain the remarkably simple recurrence formula (19). □

Theorem 2. *Under the assumptions of Theorem* 1, *the explicit expression of the integrals* $I(n,k,l)$ *in* (18) *is*

$$I(n,k,l)=(-1)^n n!A_{n,k}I(0,k+n,l)+[\tau'+(l-1)\sigma'']\varkappa_{n,k}\sum_{j=0}^{n-1}\frac{A_{n-j,k+j}}{A_{n-j-1,k+j+1}}, \tag{22}$$

where the constants $A_{n,k}$ *and* $\varkappa_{n,k}$ *are defined in Eqs.* (12) *and* (17), *respectively.*

Proof. In terms of the functions

$$f(n,k,l)=\frac{(-1)^n I(n,k,l)}{n!A_{n,k}},\qquad g(n,k,l)=\frac{(-1)^n[\tau'+(l-1)\sigma'']\varkappa_{n,k}}{n!A_{n-1,k+1}}, \tag{23}$$

Eq. (19) reads

$$f(n,k,l)=f(n-1,k+1,l)+g(n,k,l).$$

By repeated use of this recurrence relation, we readily find the explicit formula

$$f(n,k,l)=f(0,k+n,l)+\sum_{j=0}^{n-1}g(n-j,k+j,l),$$

which recalling Eq. (23) can be written in terms of the original integral $I(n,k,l)$ as,

$$I(n,k,l)=(-1)^n n!A_{n,k}\left\{I(0,k+n,l)+\sum_{j=0}^{n-1}g(n-j,k+j,l)\right\}$$

$$=(-1)^n n!A_{n,k}\left\{I(0,k+n,l)+[\tau'+(l-1)\sigma'']\sum_{j=0}^{n-1}\frac{(-1)^{n-j}\varkappa_{n-j,k+j}}{(n-j)!A_{n-j-1,k+j+1}}\right\}$$

$$=(-1)^n n!A_{n,k}\left\{I(0,k+n,l)+[\tau'+(l-1)\sigma'']\gamma_{n+k}\sum_{j=0}^{n-1}\frac{A_{n-j,k+j}}{A_{n-j-1,k+j+1}}\right\}.$$

Finally, using Eq. (17), we can further simplify this expression for $I(n,k,l)$ to that given in Eq. (22), which completes the proof of Theorem 2. □

Eq. (22) provides us with a closed analytical expression for $I(n,k,l)$ in terms of the much simpler integrals

$$I(0,k+n,l) = -\int_a^b \omega_{k+n}(x) \ln \omega_l(x)\, \mathrm{d}x. \tag{24}$$

In the particular case $k=l=0$, we immediately obtain from Theorem 2 a closed analytical expression for the integrals I_n.

Corollary 3. *Under the assumptions of Theorem* 1, *the explicit expression of the integrals* I_n *in* (6) *is*

$$I_n = (-1)^n n! A_{n,0} I(0,n,0) + (\tau' - \sigma'')\varkappa_n \sum_{j=0}^{n-1} \frac{A_{n-j,j}}{A_{n-j-1,j+1}}, \tag{25}$$

where

$$I(0,n,0) = -\int_a^b \omega_n(x) \ln \omega(x)\, \mathrm{d}x,$$

and the constants $A_{n,0}$ *and* $\varkappa_n = \varkappa_{n,0}$ *are defined in Eqs.* (12) *and* (15), *respectively.*

4. Application to the classical families

In this section we shall apply Eq. (22) and its particular case (25) to find the values of the integrals $I(n,k,l)$ and I_n for the three classical families of monic polynomials, orthogonal on the real axis (Hermite, Laguerre and Jacobi). All the necessary data concerning these families of polynomials (see e.g., [15]) are gathered in Table 1, which we shall often make use of without explicit reference to it.

4.1. Hermite polynomials

For the monic Hermite polynomials $H_n(x)$, the integrals $I(0,k+n,l)$ in (24) are readily evaluated as

$$I(0,k+n,l) = \int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^2}\, \mathrm{d}x = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}. \tag{26}$$

Hence, a simple calculation using (22) yields

$$I(n,k,l) = 2^{-n} n! \sqrt{\pi}(n + \tfrac{1}{2}). \tag{27}$$

Table 1

$p_n(x)$	Hermite $H_n(x)$	Laguerre $L_n^{(\alpha)}(x)$	Jacobi $P_n^{(\alpha,\beta)}(x)$
(a,b)	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$\alpha+1-x$	$\beta-\alpha-(\alpha+\beta+2)x$
$\omega_k(x)$	e^{-x^2}	$x^{\alpha+k}\mathrm{e}^{-x}$	$(1-x)^{\alpha+k}(1+x)^{\beta+k}$
γ_k	$\sqrt{\pi}$	$\Gamma(k+\alpha+1)$	$\dfrac{2^{\alpha+\beta+2k+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(2k+\alpha+\beta+2)}$
$A_{n,k}$	$(-2)^{-n}$	$(-1)^n$	$\dfrac{(-1)^n\Gamma(n+2k+\alpha+\beta+1)}{\Gamma(2n+2k+\alpha+\beta+1)}$
$p_{n,k}(x)$	$H_n(x)$	$L_n^{(\alpha+k)}(x)$	$P_n^{(\alpha+k,\beta+k)}(x)$

We see that in this case $I(n,k,l)$ turns out to be independent of k and l, so that, in particular, $I_n = I(n,k,l)$ (cf. [21,26]).

4.2. Laguerre polynomials

For the monic Laguerre polynomials of parameter α, $L_n^{(\alpha)}(x)$, we have

$$I(0,k+n,l) = -(\alpha+l)\int_0^\infty x^{\alpha+k+n}\mathrm{e}^{-x}\ln x\,\mathrm{d}x + \int_0^\infty x^{\alpha+k+n+1}\mathrm{e}^{-x}\,\mathrm{d}x. \qquad (28)$$

Using the well-known integral definition of the Gamma function,

$$\int_0^\infty x^{z-1}\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(z),$$

and the derivative of this equation with respect to z,

$$\int_0^\infty x^{z-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x = \Gamma(z)\psi(z),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$, Eq. (28) reads

$$I(0,k+n,l) = \Gamma(\alpha+k+n+1)[-(\alpha+l)\psi(\alpha+k+n+1)+\alpha+k+n+1].$$

After some algebra, (22) then leads to

$$I(n,k,l) = n!\Gamma(\alpha+k+n+1)[-(\alpha+l)\psi(\alpha+k+n+1)+2n+k+\alpha+1] \qquad (29)$$

and, in the particular case $k=l=0$, we find that (cf. [13,26]),

$$I_n = n!\Gamma(\alpha+n+1)[-\alpha\psi(\alpha+n+1)+2n+\alpha+1]. \qquad (30)$$

4.3. Jacobi polynomials

For the monic Jacobi polynomials of parameters α and β, $P_n^{(\alpha,\beta)}(x)$, we have

$$I(0,k+n,l) = -(\alpha+l)\int_{-1}^{1}(1-x)^{\alpha+k+n}(1+x)^{\beta+k+n}\ln(1-x)\,\mathrm{d}x$$

$$-(\beta+l)\int_{-1}^{1}(1-x)^{\alpha+k+n}(1+x)^{\beta+k+n}\ln(1+x)\,\mathrm{d}x. \tag{31}$$

Another well-known integration formula states that

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\,\mathrm{d}x = (b-a)^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

Differentiating this equation with respect to the parameters μ and ν we obtain, respectively,

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\ln(x-a)\,\mathrm{d}x = (b-a)^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

$$\times[\ln(b-a)+\psi(\mu)-\psi(\mu+\nu)],$$

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\ln(b-x)\,\mathrm{d}x = (b-a)^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

$$\times[\ln(b-a)+\psi(\nu)-\psi(\mu+\nu)].$$

Applying these formulas with $a=-1$, $b=1$, $\mu=n+k+\beta+1$, $\nu=n+k+\alpha+1$, we obtain from (31),

$$I(0,k+n,l) = \frac{2^{2n+2k+\alpha+\beta+1}\,\Gamma(n+k+\alpha+1)\,\Gamma(n+k+\beta+1)}{\Gamma(2n+2k+\alpha+\beta+2)}$$

$$\times\{-(\alpha+l)[\ln 2+\psi(n+k+\alpha+1)-\psi(2n+2k+\alpha+\beta+2)]$$

$$-(\beta+l)[\ln 2+\psi(n+k+\beta+1)-\psi(2n+2k+\alpha+\beta+2)]\}.$$

A somewhat lengthy but straightforward calculation using (22) then leads to

$$I(n,k,l) = \frac{2^{2n+2k+\alpha+\beta+1}\,n!\,\Gamma(n+k+\alpha+1)\,\Gamma(n+k+\beta+1)\,\Gamma(n+2k+\alpha+\beta+1)}{(2n+2k+\alpha+\beta+1)[\Gamma(2n+2k+\alpha+\beta+1)]^2}$$

$$\times\Bigg\{-(\alpha+l)\psi(n+k+\alpha+1)-(\beta+l)\psi(n+k+\beta+1)$$

$$+(\alpha+\beta+2l)\Bigg[-\ln 2+\frac{1}{2n+2k+\alpha+\beta+1}$$

$$+2\psi(2n+2k+\alpha+\beta+1)-\psi(n+2k+\alpha+\beta+1)\Bigg]\Bigg\}, \tag{32}$$

where we have taken advantage of the well-known recurrence formulas for the Gamma and psi functions,

$$\Gamma(z+1)=z\Gamma(z), \quad \psi(z+1)=\psi(z)+\frac{1}{z}, \tag{33}$$

in order to simplify the final expression for $I(n,k,l)$. In the particular case $k=l=0$, Eq. (32) reduces to

$$I_n=\frac{2^{2n+\alpha+\beta+1}n!\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)[\Gamma(2n+\alpha+\beta+1)]^2}$$

$$\times\left\{-\alpha\psi(n+\alpha+1)-\beta\psi(n+\beta+1)+(\alpha+\beta)\left[-\ln 2+\frac{1}{2n+\alpha+\beta+1}\right.\right.$$

$$\left.\left.+2\psi(2n+\alpha+\beta+1)-\psi(n+\alpha+\beta+1)\right]\right\}. \tag{34}$$

In the case when $\alpha=\beta=\lambda-\frac{1}{2}$, we obtain from Eqs. (32) and (34) the expressions of $I(n,k,l)$ and I_n for the monic Gegenbauer polynomials of parameter λ, $C_n^{(\lambda)}(x)=P_n^{(\lambda-1/2,\lambda-1/2)}(x)$,

$$I(n,k,l)=\frac{(2\lambda-1+2l)\pi n!\,\Gamma(n+2k+2\lambda)}{2^{2(n+k+\lambda)-1}(n+k+\lambda)[\Gamma(n+k+\lambda)]^2}$$

$$\times\left[\frac{1}{2(n+k+\lambda)}+\ln 2+\psi(n+k+\lambda)-\psi(n+2k+2\lambda)\right],$$

$$I_n=\frac{(2\lambda-1)\pi n!\,\Gamma(n+2\lambda)}{2^{2(n+\lambda)-1}(n+\lambda)[\Gamma(n+\lambda)]^2}$$

$$\times\left[\frac{1}{2(n+\lambda)}+\ln 2+\psi(n+\lambda)-\psi(n+2\lambda)\right], \tag{35}$$

where (33) and the duplication formulas for the Gamma and psi functions,

$$\Gamma(2z)=\frac{2^{2z-1}\,\Gamma(z+1/2)\Gamma(z)}{\sqrt{\pi}}, \qquad \psi(2z)=\tfrac{1}{2}\psi(z)+\tfrac{1}{2}\psi(z+\tfrac{1}{2})+\ln 2,$$

have been used to further simplify the expressions. The result obtained for I_n agrees with that given in [22,26,27], after taking into account the different normalizations used for the Gegenbauer polynomials.

5. Summary

The examples above illustrate our general method to compute the integrals I_n defined by Eq. (6), which appear in the calculation of the Boltzmann–Shannon entropies of probability measures involving continuous orthogonal polynomials of hypergeometric type with general supports. This method makes use only of the coefficients of the differential operator corresponding to the hypergeometric polynomials, and enables us to obtain both a recursion relation and an explicit expression for the family of integrals defined by (18), which includes I_n as a particular case. It is worth noting that similar approaches have been recently applied to the calculation of connection and linearization coefficients [4,19] and to the derivation of recurrence formulas of several kinds for hypergeometric-type functions [11,17,24].

Acknowledgements

This work has been partially supported by the Junta de Andalucía, under the research grant FQM0207, and by the Spanish DGES projects PB96-0170 (J.S.R.) and PB95-1205. Also, partial support from the European project INTAS-93-219-EXT is acknowledged.

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